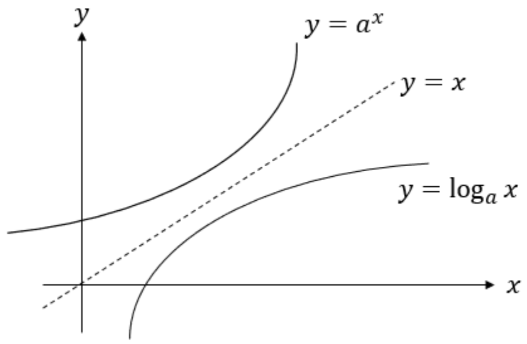
Mathematics Methods

Unit 4

Logarithm

1. Introduction

Logarithm is the inverse function to exponentiation.



$$\boxed{y = \log_a x}$$

Condition

- $a > 0, \ a \neq 1$
- x cannot be negative

Representation of logarithm

Notation	Logarithm	Uses
$lg\ x$ $log\ x$	$\log_{10} x$ Logarithm base "10"	Various engineering fields, logarithm tables, handheld calculators, spectroscopy
$ln\ x$	$\log_e x$ Logarithm base "e"	Mathematics, physics, chemistry, statistics, economics, information theory, and engineering
$lb\ x$	$\log_2 x$ Logarithm base "2"	Computer science, information theory, music theory, photography

2. Relationship between logarithm and exponentiation

Relationship between logarithm and exponentiation:

$$y = \log_z x \text{ if and only if } x = z^y$$

Example:

Exponential form is $144 = 12^2$ while logarithmic form is $\log_{12} 144 = 2$

Example 1:

Rewrite $\log_{10} 10000$ in exponential form and solve this equation.

Let $\log_{10} 10000 = x$

$$10000 = 10^x$$

$$10^4 = 10^x$$

$$\Rightarrow \quad x = 4$$

$\therefore \log_{10} 10000 = 4$

Example 2:

If $\frac{\log k}{g-h} = \frac{\log g}{h-k} = \frac{\log h}{k-g}$, evaluate $k^{g+h} \times g^{h+k} \times h^{k+g}$.

Let $\frac{\log k}{g-h} = \frac{\log g}{h-k} = \frac{\log h}{k-g} = w$,

$$\frac{\log k}{g-h} = w \qquad \frac{\log g}{h-k} = w \qquad \frac{\log h}{k-g} = w$$

$$\log k = w(g-h) \qquad \log g = w(h-k) \qquad \log h = w(k-g)$$

$$k = 10^{w(g-h)} \qquad g = 10^{w(h-k)} \qquad h = 10^{w(k-g)}$$

$$\begin{aligned} k^{g+h} \times g^{h+k} \times h^{k+g} &= 10^{w(g-h)(g+h)} \times 10^{w(h-k)(h+k)} \times 10^{w(k-g)(k+g)} \\ &= 10^{w(g^2+hg-hg-h^2)} \times 10^{w(h^2+hk-hk-k^2)} \times 10^{w(k^2+gk-gk-b^2)} \\ &= 10^{w(g^2-h^2+h^2-k^2+k^2-g^2)} \\ &= 10^{w(0)} \\ &= 10^0 \\ &= 1 \end{aligned}$$

Example 3:

Solve $\log_{2^2}(z+1) = 2$ by rewriting it in exponential form.

$$\begin{aligned} \log_{2^2}(z+1) &= 2 \\ z+1 &= 4^2 \\ z &= 15 \end{aligned}$$

3. Algebraic properties of logarithm

Properties:

$\log_x x^y = y$ or $x^{\log_x y} = y$

$\log_x 1 = 0$

$\log_x x = 1$

<u>Change of base</u>

$$\log_b c = \frac{\log_a c}{\log_a b}$$

Example 1:

Solve $\log_2(x+2) + \log_2 2 = 4$

$$\begin{aligned} \log_2(x+2) + \log_2 2 &= 4 \\ \log_2(x+2) + 1 &= 4 \\ \log_2(x+2) &= 3 \\ x+2 &= 2^3 \\ x+2 &= 8 \\ x &= 6 \end{aligned}$$

Example 2:

Solve $0.16^{\log_{\frac{2}{5}}[\frac{1}{3}+\frac{1}{3^2}+\ldots]}$.

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{\frac{1}{3}}{1-\frac{1}{3}}$$
$$= \frac{1}{2}$$

$$0.16^{\log_{\frac{2}{5}}[\frac{1}{3}+\frac{1}{3^2}+\ldots]} = 0.16^{\log_{\frac{2}{5}}\frac{1}{2}}$$
$$= (\frac{2}{5})^{2\log_{\frac{2}{5}}\frac{1}{2}}$$
$$= (\frac{2}{5})^{\log_{\frac{2}{5}}(\frac{1}{2})^2}$$
$$= (\frac{2}{5})^{\log_{\frac{2}{5}}(\frac{1}{4})}$$
$$= \frac{1}{4}$$

Example 3:

If $x = \log_{2a} a$, $y = \log_{3a} 2a$, $z = \log_{4a} 3a$, prove that $1 + xyz = 2yz$.

[Note that $\log_a xy = \log_a x + \log_a y$]

$$1 + xyz = 2yz$$

LHS,

$1 + xyz$

$= 1 + (\log_{2a} a)(\log_{3a} 2a)(\log_{4a} 3a)$

$= \log_{4a} 4a + (\log_{2a} a)(\frac{\log_{2a} 2a}{\log_{2a} 3a})(\log_{4a} 3a)$

$= \log_{4a} 4a + \frac{\log_{2a} a}{\log_{2a} 3a}(\log_{4a} 3a)$

$= \log_{4a} 4a + \frac{\frac{\log_{3a} a}{\log_{3a} 2a}}{\frac{\log_{3a} 3a}{\log_{3a} 2a}}(\log_{4a} 3a)$

$= \log_{4a} 4a + \frac{\log_{3a} a}{\log_{3a} 2a}\left(\frac{\log_{3a} 2a}{\log_{3a} 3a}\right)(\log_{4a} 3a)$

$= \log_{4a} 4a + \log_{3a} a\,(\frac{\log_{3a} 3a}{\log_{3a} 4a})$

$= \log_{4a} 4a + \frac{\log_{3a} a}{\log_{3a} 4a}$

$= \log_{4a} 4a + \frac{\frac{\log_{4a} a}{\log_{4a} 3a}}{\frac{\log_{4a} 4a}{\log_{4a} 3a}}$

$= \log_{4a} 4a + \frac{\log_{4a} a}{\log_{4a} 3a}(\frac{\log_{4a} 3a}{\log_{4a} 4a})$

	$= \log_{4a} 4a + \log_{4a} a$ $= \log_{4a} 4a^2$ RHS, $2yz$ $= 2(\log_{3a} 2a)(\log_{4a} 3a)$ $= 2(\frac{\log_{4a} 2a}{\log_{4a} 3a})\ (\log_{4a} 3a)$ $= 2(\log_{4a} 2a)$ $= \log_{4a} 4a^2$ LHS = RHS (proven)
4.	**Laws of logarithm**
	(a) Product law
	Formula: $$\log_a xy = \log_a x + \log_a y$$ Derivation of formula: Suppose $\log_a x = b$ and $\log_a y = c$ $a^b(a^c) = a^{b+c}$ $xy = a^{b+c}$ $\log_a xy = b + c$ Substitute $\log_a x = b$ and $\log_a y = c$, $\log_a xy = \log_a x + \log_a y$
	Example 1: If $\log_3 2 = v$ and $\log_3 5 = q$. Express $\log_3 10$ in terms of v and q. $\log_3 10 = \log_3(2 \times 5)$ $= \log_3 2 + \log_3 5$ $= v + q$ Example 2: Express $\log_{45} k + \log_{45} j$ as a single logarithm. $$\log_{45} k + \log_{45} j = \log_{45} kj$$ Example 3: Express y in terms of x for $\log_{\sqrt{2}} y = \log_{\sqrt{2}} x^2 + \log_{\sqrt{2}} 2\sqrt{5}$. $\log_{\sqrt{2}} y = \log_{\sqrt{2}} x^2 + \log_{\sqrt{2}} 2\sqrt{5}$ $y = 2\sqrt{5}x^2$

(b) Quotient law

Formula:

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

Derivation of formula:
Suppose $\log_a x = b$ and $\log_a y = c$

$$\frac{a^b}{a^c} = a^{b-c}$$
$$\frac{x}{y} = a^{b-c}$$
$$\log_a \frac{x}{y} = b - c$$

Substitute $\log_a x = b$ and $\log_a y = c$,
$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

Example 1:
Express $\log_{3\sqrt{\frac{1}{k}}} 3 - \log_{3\sqrt{\frac{1}{k}}} 6$ as a single logarithm.

$$\log_{3\sqrt{\frac{1}{k}}} 3 - \log_{3\sqrt{\frac{1}{k}}} 6 = \log_{3\sqrt{\frac{1}{k}}} \frac{3}{6}$$
$$= \log_{3\sqrt{\frac{1}{k}}} \frac{1}{2}$$

Example 2:
If $\log_r 3 = k$, $\log_r 2 = l$ and $\log_r 12 = m$. Express $\log_r 2$ in terms of k, l and m.

$$\log_r 2 = \log_r (\frac{12}{3 \times 2})$$
$$= \log_r 12 - (\log_r 3 + \log_r 2)$$
$$= m - k - l$$

(c) Power law

Formula:

$$\log_a x^n = n \log_a x$$

Derivation of formula:
Suppose $\log_a x = b$ and $\log_a y = c$

$(a^b)^n = a^{bn}$
Substitute $x = a^b$,
$x^n = a^{bn}$
$\log_a x^n = bn$

Substitute $\log_a x = b$,
$\log_a x^n = n \log_a x$

6.	**Solving logarithmic functions**
	Summary of several tips to solve logarithmic functions: 1. Convert to index form 2. Use quadratic equation/ substitute variable 3. Comparison method
	(a) Converting to index form
	Example 1: Solve $\log_2(x+1) + \log_2 4 = 2$. $\log_2(x+1) + \log_2 4 = 2$ $\log_2(x+1) + 2 = 2$ $\log_2(x+1) = 0$ $x + 1 = 2^0$ $x = 1 - 1$ $x = 0$ Example 2: Solve $\log_x 25 + \log_x 100 - 2 = 0$. $\log_x 25 + \log_x 100 - 2 = 0$ $\log_x 25 + \log_x 100 = 2$ $\log_x(25 \times 100) = 2$ $2500 = x^2$ $x = \sqrt{2500}$ $= 50$ Example 3: Solve $\log_{\sqrt[3]{r^2}} 2 + \log_{\sqrt[3]{r^2}} k = 3$. Express your answer in terms of r. $\log_{\sqrt[3]{r^2}} 2 + \log_{\sqrt[3]{r^2}} k = 3$ $\log_{\sqrt[3]{r^2}}(2k) = 3$ $2k = \sqrt[3]{r^2}^3$ $2k = r^{\frac{2}{3}(3)}$ $2k = r^2$ $k = \frac{r^2}{2}$
	(b) Using quadratic equation/ substituting variable
	Example 1: Solve $[\log_5 x]^2 + \log_5 x - 1 = 0$. $[\log_5 x]^2 + \log_5 x - 1 = 0$ Let $y = \log_5 x$, $y^2 + y - 1 = 0$ $y = 0.618$ or $y = -1.618$ $\log_5 x = 0.618$ $\log_5 x = -1.618$ $x = 5^{0.618}$ $x = 5^{-1.618}$ $= 2.704$ $= 0.074$

Example 2:
Solve $2[\log_x 2]^2 = \log_x 128 - 3$.

$$2[\log_x 2]^2 = 7\log_x 2 - 3$$

Let $y = \log_x 2$,

$$2y^2 = 7y - 3$$
$$y = \frac{1}{2} \quad or \quad y = 3$$
$$\log_x 2 = \frac{1}{2} \qquad \log_x 2 = 3$$
$$x^{\frac{1}{2}} = 2 \qquad x^3 = 2$$
$$x = 4 \qquad x = \sqrt[3]{2}$$

Example 3:
Solve $[\log_{10} x]^3 - [\frac{1}{2}\log_{10} x^2]^2 - \log_{10} x + 1 = 0$.

$$[\log_{10} x]^3 - [\frac{1}{2}\log_{10} x^2]^2 - \log_{10} x + 1 = 0$$

Let $y = \log_{10} x$,

$$y^3 - y^2 - y + 1 = 0$$
$$y = 1 \quad \text{or} \quad y = -1$$
$$\log_{10} x = 1 \qquad \log_{10} x = -1$$
$$x = 10^1 \qquad x = 10^{-1}$$
$$= 10 \qquad = \frac{1}{10}$$

(c) Comparison method

Example 1:
Solve $\log_3 2 + \log_3(x-1) = 3^a$. Express your answer in term of x.

$$\log_3 2 + \log_3(x-1) = \log_3 3^{3^a}$$
$$\log_3 2(x-1) = \log_3 3^{3^a}$$
$$2x - 2 = 3^{3^a}$$
$$2x = 3^{3^a} + 2$$
$$x = \frac{3^{3^a} + 2}{2}$$
$$= \frac{3^{3^a}}{2} + 1$$

Example 2:
Solve $\log_{k^2} r + \log_{k^2} 2d = 1$ given that $r(2d) = 4$.

$$\log_{k^2} r + \log_{k^2} 2d = 1$$
$$\log_{k^2} r + \log_{k^2} 2d = \log_{k^2} k^2$$
$$\log_{k^2} r\,(2d) = \log_{k^2} k^2$$
$$r(2d) = k^2$$
$$k^2 = 4$$
$$k = \pm\sqrt{4}$$
$$k = 2 \quad \text{or} \quad k = -2$$

Example 3:
Solve $\log_2(x+2) + \log_2 2 = 2$.

$$\begin{aligned} \log_2(x+2) + \log_2 2 &= 2 \\ \log_2 2(x+2) &= \log_2 2^2 \\ 2x + 4 &= 4 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

5. Using logarithm to solve exponential problems

Example 1:
Solve $2^{x+1} = 7^{2x-1}$.

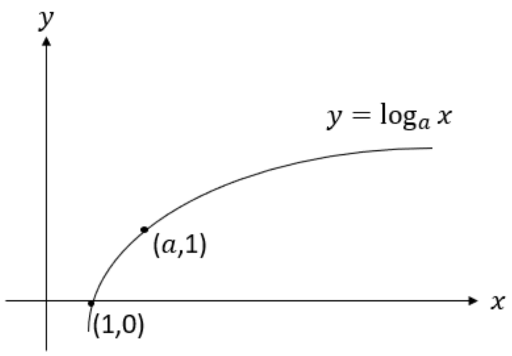
$$\begin{aligned} 2^{x+1} &= 7^{2x-1} \\ \ln 2^{x+1} &= \ln 7^{2x-1} \\ (x+1)\ln 2 &= (2x-1)\ln 7 \\ x\ln 2 + \ln 2 &= 2x\ln 7 - \ln 7 \\ x\ln 2 - 2x\ln 7 &= -\ln 7 - \ln 2 \\ x(\ln 2 - 2\ln 7) &= -\ln 7 - \ln 2 \\ x &= \frac{-\ln 7 - \ln 2}{\ln 2 - 2\ln 7} \\ &= 0.825 \end{aligned}$$

Example 2:
Solve $3^x = 4^{2x}$.

$$\begin{aligned} 3^x &= 4^{2x} \\ x\ln 3 &= 2x\ \ln 4 \\ x(\ln 3 - 2\ln 4) &= 0 \\ x &= 0 \end{aligned}$$

6. Graph sketching of logarithmic functions

General graph of logarithmic function



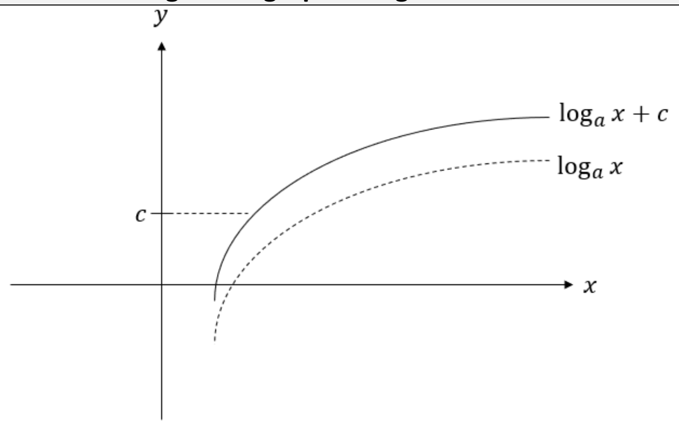
Condition: $a > 1$

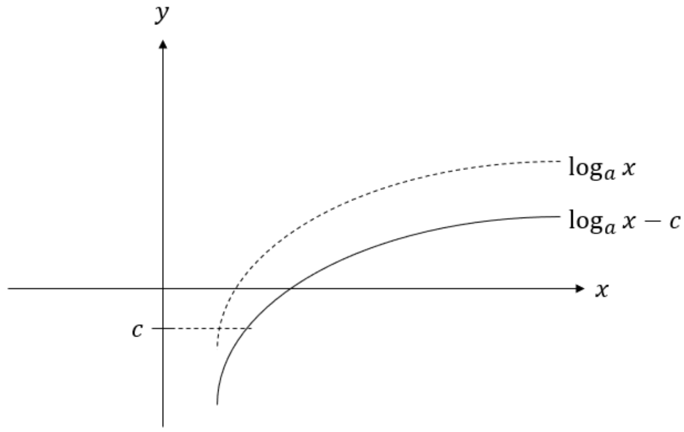
Main features/ characteristics

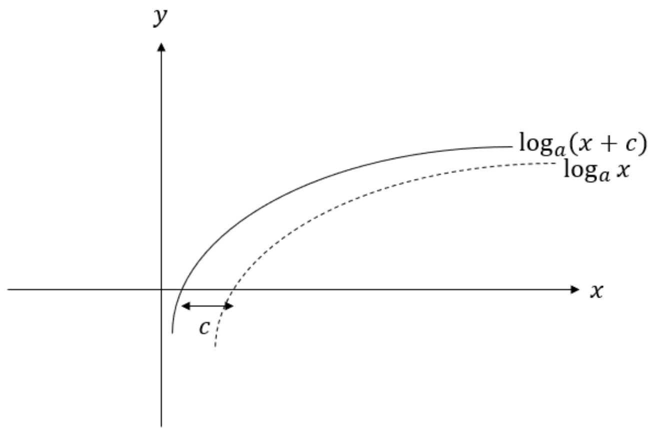
- Graph is continuous
- Domain is for $x > 0$
- Range is for all real numbers
- Vertical asymptote $x = 0$
- x-intercept at (1,0)

(a) Effect of $\pm c$

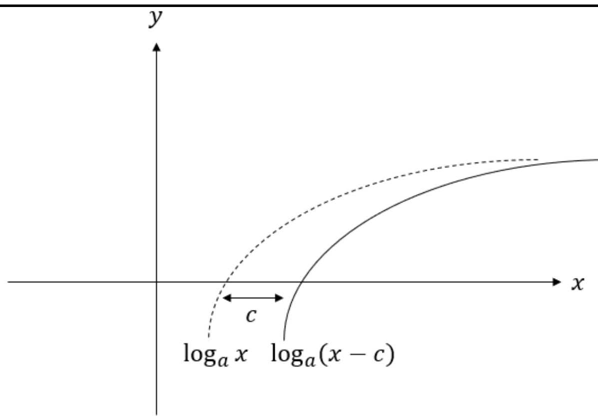
Given that the general graph of logarithmic functions is $y = \log_a x$:

<table>
<tr><th>$\log_a x$</th><th>Effect on general graph of logarithmic function</th></tr>
<tr><td>$\log_a x + c$</td><td>

Graph shifts upwards by c units</td></tr>
<tr><td>$\log_a x - c$</td><td>

Graph shifts downwards by c units</td></tr>
<tr><td>$\log_a (x + c)$</td><td>

Graph shifts leftwards by c units</td></tr>
</table>

$\log_a(x-c)$	 Graph shifts rightwards by c units

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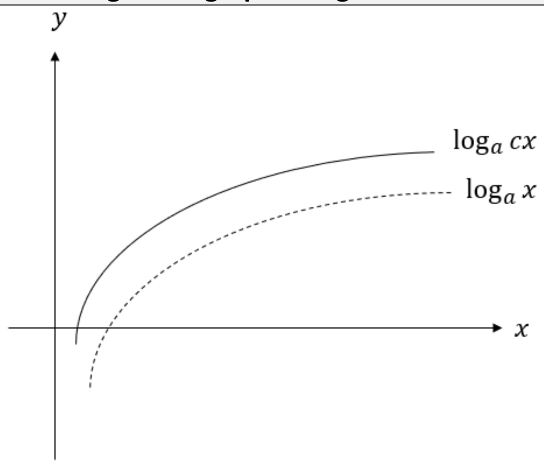
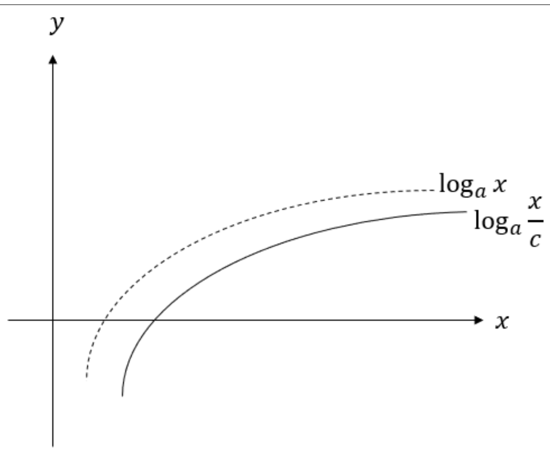
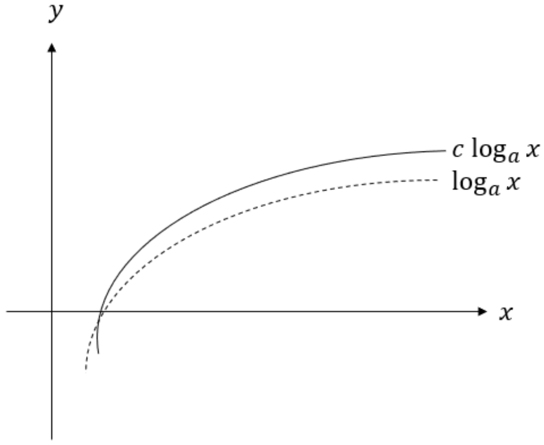
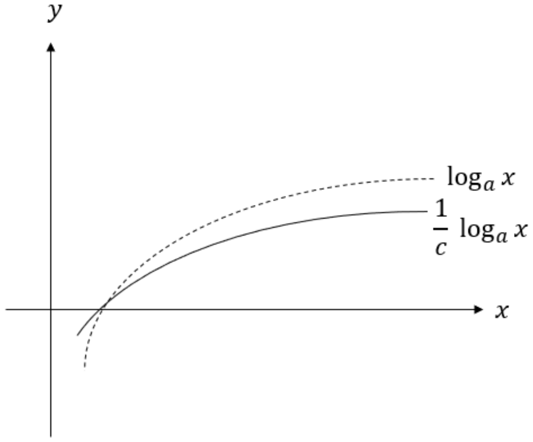
(b) Effect of $\times/\div$ c

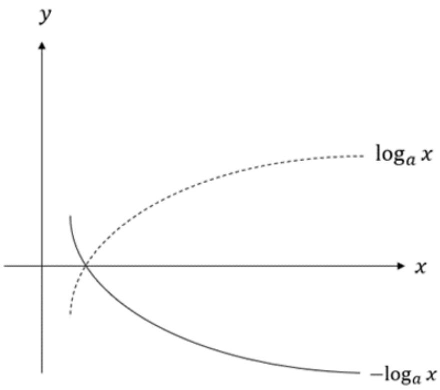
Given that the general graph of logarithmic functions is $y = \log_a x$:

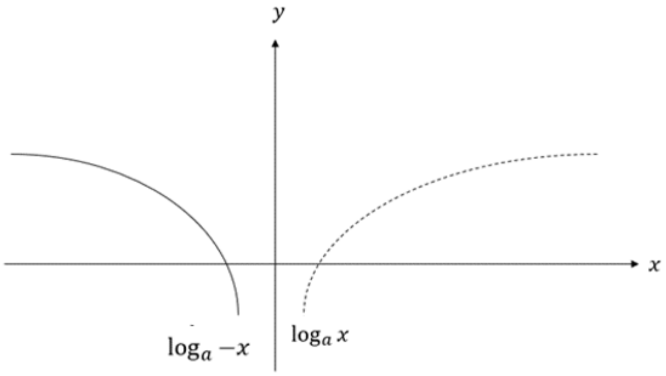
$\log_a x$	Effect on general graph of logarithmic function
$\log_a cx$ $c > 1$	 Graph compress/ shrunk horizontally by a factor of c
$\log_a \frac{x}{c}$ $c > 1$	 Graph stretch horizontally by a factor of c

$c \log_a x$ $c > 1$	 Graph stretch vertically by a factor of c
$\frac{1}{c}\log_a x$ $c > 1$	 Graph compress/ shrunk vertically by a factor of c

(c) Effect of " $-$ "

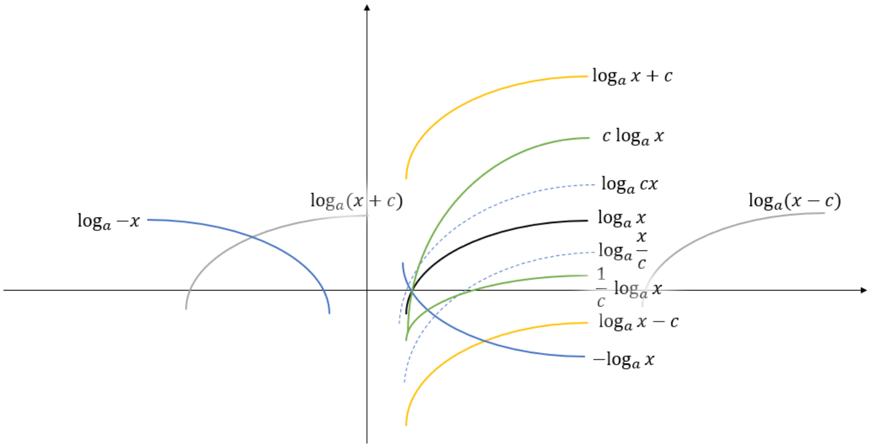
Given that the general graph of logarithmic functions is $y = \log_a x$:

$\log_a x$	Effect on general graph of logarithmic function
$-\log_a x$	 Graph reflects about $x = 0$/ x $-$axis

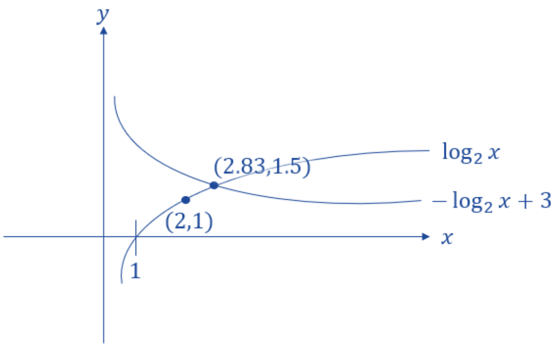
$\log_a -x$	 Graph reflects about $y = 0$/ y-axis

Summary:

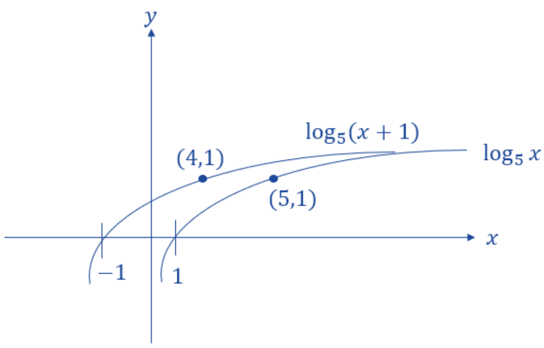
Transformation	Effect
$\log_a x + c$	Graph shifts upwards by c units
$\log_a x - c$	Graph shifts downwards by c units
$\log_a (x + c)$	Graph shifts leftwards by c units
$\log_a (x - c)$	Graph shifts rightwards by c units
$\log_a cx$ $c > 1$	Graph compress/ shrunk horizontally by a factor of c
$\log_a \frac{x}{c}$ $c > 1$	Graph stretch horizontally by a factor of c
$c \log_a x$ $c > 1$	Graph stretch vertically by a factor of c
$\frac{1}{c} \log_a x$ $c > 1$	Graph compress/ shrunk vertically by a factor of c
$-\log_a x$	Graph reflects about $x = 0$/ x –axis
$\log_a -x$	Graph reflects about $y = 0$/ y-axis



Example 1:
Sketch the graph of $\log_2 x$ and $-\log_2 x + 3$ in the same axis showing important points and features.



Example 2:
Sketch the graph of $\log_5 x$ and $\log_5(x + 1)$ in the same axis showing important points and features.



7.	**Applications of logarithm**
	(a) Richter scale

Formula:

$$R = \log_{10} I_R$$

$I_R = \frac{I_c}{I_n}$,

$$R = \log_{10} \frac{I_c}{I_n}$$

or

$$R = 0.67(\log 0.37E) + 1.46$$

I_R: Relative intensity of earthquake compared to the smallest tremor felt by humans
I_c: Intensity of earthquake *(measured by the amplitude of a seismograph reading taken 100km from the epicentre of the earthquake)*
I_n: Intensity of standard earthquake with amplitude 1 micron= 10^{-4} cm
E: Energy released by earthquake in kW

Example 1:
If an earthquake is 24.5 times more intense, how much larger is its magnitude on the Richter scale?

$$R_1 = \log_{10} \frac{24.5I}{I_0}$$
$$R_2 = \log_{10} \frac{I}{I_0}$$

$$R_1 - R_2 = \log_{10} \frac{24.5I}{I_0} - \log_{10} \frac{I}{I_0}$$
$$= \log_{10}\left(\frac{24.5I}{I_0} \div \frac{I}{I_0}\right)$$
$$= \log_{10}(24.5)$$
$$= 1.39 \text{ times} \quad \textit{(2 decimal places)}$$

Example 2:
City X had experienced an earthquake which releases $1.2 \times 10^7 kW$ of energy. Calculate the magnitude of the earthquake.

$$R = 0.67(\log 0.37E) + 1.46$$
$$= 0.67[\log 0.37(1.2 \times 10^7)] + 1.46$$
$$= 5.9$$

(b) pH scale

Formula:

$$pH = -\log_{10} H^+$$

or

$$pH = -\log_{10} H_3O^+$$

H^+: Concentration of hydrogen ions in solution $(mol/\ \ell)$
H_3O^+: Concentration of hydronium ions in solution $(mol/\ \ell)$

Example 1:
A solution has $1 \times 12^2\ M$ of hydrogen ions. Calculate it's pH value.

$$\begin{aligned} pH &= -\log_{10} H^+ \\ &= -\log_{10} 1 \times 12^2 \\ &= -2.16 \end{aligned}$$

Example 2:
Calculate the concentration of hydronium ion concentration, H_3O^+ in a juice of 3.5 pH.

$$\begin{aligned} pH &= -\log_{10} H_3O^+ \\ 3.5 &= -\log_{10} H_3O^+ \\ 10^{-3.5} &= H_3O^+ \\ H_3O^+ &= 3.16 \times 10^{-4}\ mol/\ \ell \end{aligned}$$

(c) Loudness scale

Formula:

$$L = 10 \log_{10}(\frac{I}{I_o})$$

I_o: reference sound/ threshold of hearing $= 10^{-12}\ W/m^2$

Example 1:
The intensity level of a concert is 1.2 W/m^2. What is the decibel level of the concert?

$$\begin{aligned} L &= 10 \log_{10}(\frac{1.2}{10^{-12}}) \\ &= 130.79\ dB \end{aligned}$$

Example 2:
Find the ratio of sound intensity for the sound level of 50 dB and 80 dB.

$$50 = 10 \log_{10}(\frac{I_1}{I_o})$$

$$80 = 10 \log_{10}(\frac{I_2}{I_o})$$

$$80 - 50 = 10\log_{10}(\frac{I_2}{I_o}) - 10\log_{10}(\frac{I_1}{I_o})$$

$$30 = 10\log_{10}(\frac{I_2}{I_o} \div \frac{I_1}{I_o})$$

$$3 = \log_{10}(\frac{I_2}{I_1})$$

$$\frac{I_2}{I_1} = 10^3$$

(d) Music scale

Formula:

Ratio of frequencies

$$x = \log_2 \frac{f_2}{f_1}$$

$$r = \frac{f_2}{f_1} = 2^x$$

$$log(\frac{f_2}{f_1}) = x\,log\,2$$

Example 1:
An octave is a type of musical interval, or measure of distance between two notes. Specifically, an octave is the distance between one note and another note with the same letter name. What is the difference between the octave frequency and the base frequency in a piano?

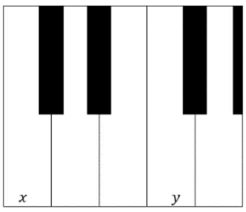
$$x = \log_2 \frac{f_2}{f_1}$$

$$= \log_2 \frac{2f}{f}$$

$$= \log_2 2$$

$$= 1$$

Example 2:
Given that a piano octave is divided into 12 notes and intervals f_o to f_{12}, with successive frequency ratios all equal to a number r. Given that $f_{12} = 2f_o$ show that $r = 2^{\frac{1}{12}}$ which is the ratio of frequency of one note to the previous. Hence, find the frequency of a note y as shown below given that note x has the frequency of $261.63Hz$.



$$\frac{f_{12}}{f_o} = r^{12}$$
$$\frac{2f_o}{f_o} = r^{12}$$
$$2 = r^{12}$$
$$\log_2 2 = 12 \log_2 r$$
$$\log_2 r = \frac{1}{12}$$
$$r = 2^{\frac{1}{12}}$$

Frequency of note y,

$f = 261.63 \times (2^{\frac{1}{12}})^5$
$= 349.234$

(e) f-stop setting of lens

Formula:

Number of stops between two stop numbers

$$S_n = (\sqrt{2})^n S_o$$
$$\frac{S_n}{S_o} = (\sqrt{2})^n$$
$$\log_{\sqrt{2}}(\frac{S_n}{S_o}) = \log_{\sqrt{2}}\sqrt{2}^n$$
$$n = \log_{\sqrt{2}}(\frac{S_n}{S_o})$$

n: Number of stops
S_n: Stop number
S_o: Initial reference stop number

Example 1:
What is the number of stops between f/5.6 and f/3.5?

$n = \log_{\sqrt{2}}\left(\frac{S_n}{S_o}\right)$
$= \log_{\sqrt{2}}\left(\frac{5.6}{3.5}\right)$
$= 1.356$
≈ 1.36 stops

Example 2:
Find the stop number given that the number of stops from the initial reference stop number, f/3.5 is 6.7.

$S_n = (\sqrt{2})^n S_o$
$= (\sqrt{2})^{6.7} 3.5$
$= 35.68$

END